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On the Conditional Probability Density Functions of Multivariate Uniform Random Vectors and Multivariate Normal Random Vectors

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It is shown that the conditional probability density function of $\mathbf{Y}_1$ given $(1/n)\sum_{i=1}^n \mathbf{Y}_i\mathbf{Y}_i' = \mathbf{\Sigma}$, where $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ are i.i.d. *p-variate uniform* random vectors with mean $\mathbf{0}$ equals to that of $\mathbf{Y}_1$ given $(1/n)\sum_{i=1}^n \mathbf{Y}_i\mathbf{Y}_i' = \mathbf{\Sigma}$, where $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ are i.i.d. *p-variate normal* random vectors with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$.

1. Introduction

The purpose of this note is to present the following *finite sample property* of the conditional probability density functions (pdfs) of multivariate uniform and multivariate normal random vectors.

2. A Theorem and its Proof

Theorem. *Let* $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ *be* i.i.d. p-variate uniform *random vectors with mean* $\mathbf{0}$. *The support is any open set* $\mathbf{U}$ *in* $\mathbf{R}^p$ *such that*

$$\left\{(\mathbf{y}_1, ..., \mathbf{y}_n): \frac{1}{n}\sum_{i=1}^n \mathbf{y}_i\mathbf{y}_i' = \mathbf{\Sigma}\right\} \subset \mathbf{U} \times \cdots \times \mathbf{U}.$$

Then, the conditional pdf *of* $\mathbf{Y}_1$ *given* $(1/n)\sum_{i=1}^n \mathbf{Y}_i\mathbf{Y}_i' = \mathbf{\Sigma}$ *is the same as that of* $\mathbf{Y}_1$ *given* $(1/n)\sum_{i=1}^n \mathbf{Y}_i\mathbf{Y}_i' = \mathbf{\Sigma}$, *where* $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ *are* i.i.d. p-variate normal *random vectors with mean* $\mathbf{0}$ *and covariance matrix* $\mathbf{\Sigma}$.

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Proof. Define an *np-dimensional* vector $\mathbf{y}_*$, an $(n-1)p$-dimensional vector $\mathbf{y}_*^{(2)}$ and an $np \times np$ matrix $\mathbf{\Sigma}_*$ as

$$\mathbf{y}_* = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{pmatrix}, \qquad \mathbf{y}_*^{(2)} = \begin{pmatrix} \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_n \end{pmatrix}$$

and

$$\mathbf{\Sigma}_* = \mathbf{I}_n \otimes \mathbf{\Sigma},$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{A} \otimes \mathbf{B}$ means the Kronecker product of the matrices $\mathbf{A}$ and $\mathbf{B}$. Also, let

$$\mathbf{G} = \left\{ \mathbf{y}_* \;\middle|\; \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t = \mathbf{\Sigma} \right\}$$

and

$$\mathbf{G}_\varepsilon = \left\{ \mathbf{y}_* \;\middle|\; \left\| \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t - \mathbf{\Sigma} \right\| < \varepsilon \right\},$$

where $\|\mathbf{A}\|$ means a norm of the matrix $\mathbf{A}$. It may be any subordinate norm, or the Euclidean norm or the spectral norm.

For any $\mathbf{y}_* \in \mathbf{G}_\varepsilon$, the following holds:

$$\begin{aligned} |\mathbf{y}_*^t \mathbf{\Sigma}_*^{-1} \mathbf{y}_* - np| &= \left| \operatorname{trace}\left(\sum_{i=1}^{n} \mathbf{y}_i^t \mathbf{\Sigma}^{-1} \mathbf{y}_i \right) - np \right| \\ &= \left| \operatorname{trace}\left(\mathbf{\Sigma}^{-1} \sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t - n\mathbf{I}_p \right) \right| \\ &= \left| \operatorname{trace}\left[\mathbf{\Sigma}^{-1} \left(\sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t - n\mathbf{\Sigma} \right) \right] \right| \\ &\leqslant p \left\| \mathbf{\Sigma}^{-1} \left(\sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t - n\mathbf{\Sigma} \right) \right\| \\ &\leqslant p \, \|\mathbf{\Sigma}^{-1}\| \left\| \sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^t - n\mathbf{\Sigma} \right\| \\ &\leqslant np \, \|\mathbf{\Sigma}^{-1}\| \, \varepsilon. \end{aligned} \tag{1}$$

The conditional pdf of $\mathbf{y}_1$ given $\mathbf{G}$ is

$$f(\mathbf{y}_1 \mid \mathbf{G}) = \lim_{\varepsilon \to 0} f(\mathbf{y}_1 \mid \mathbf{G}_\varepsilon) = \lim_{\varepsilon \to 0} \frac{\int_{G_\varepsilon} \mathbf{dy}_*^{(2)}}{\int_{G_\varepsilon} \mathbf{dy}_*}.$$

The denominator of the ratio is

$$\int_{G_\varepsilon} \mathbf{dy}_* = \int_{G_\varepsilon} \exp(\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2) \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*.$$

Thus, Inequality (1) implies that

$$\exp\{np(1 - \|\mathbf{\Sigma}^{-1}\|\ \varepsilon)/2\} \leqslant \frac{\int_{G_\varepsilon} \mathbf{dy}_*}{\int_{G_\varepsilon} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*} \leqslant \exp\{np(1 + \|\mathbf{\Sigma}^{-1}\|\ \varepsilon)/2\}.$$

The following inequalities can be similarly shown:

$$\exp\{np(1 - \|\mathbf{\Sigma}^{-1}\|\ \varepsilon)/2\} \leqslant \frac{\int_{G_\varepsilon} \mathbf{dy}_*^{(2)}}{\int_{G_\varepsilon} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*^{(2)}} \leqslant \exp\{np(1 + \|\mathbf{\Sigma}^{-1}\|\ \varepsilon)/2\}.$$

These inequalities imply that

$$\exp\{-np\ \|\mathbf{\Sigma}^{-1}\|\ \varepsilon\} \leqslant \frac{\int_{G_\varepsilon} \mathbf{dy}_*}{\int_{G_\varepsilon} \mathbf{dy}_*^{(2)}} \frac{\int_{G_\varepsilon} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*^{(2)}}{\int_{G_\varepsilon} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*} \leqslant \exp\{np\ \|\mathbf{\Sigma}^{-1}\|\ \varepsilon\}.$$

Therefore, the following holds:

$$\begin{aligned} f(\mathbf{y}_1 \mid \mathbf{G}) &= \lim_{\varepsilon \to 0} f(\mathbf{y}_1 \mid \mathbf{G}_\varepsilon) = \lim_{\varepsilon \to 0} \frac{\int_{G_\varepsilon} \mathbf{dy}_*^{(2)}}{\int_{G_\varepsilon} \mathbf{dy}_*} \\ &= \lim_{\varepsilon \to 0} \frac{\int_{G_\varepsilon} (2\pi)^{-np/2} \det(\mathbf{\Sigma}_*)^{-1/2} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*^{(2)}}{\int_{G_\varepsilon} (2\pi)^{-np/2} \det(\mathbf{\Sigma}_*)^{-1/2} \exp(-\mathbf{y}_*' \mathbf{\Sigma}_*^{-1} \mathbf{y}_*/2)\, \mathbf{dy}_*} \\ &= \lim_{\varepsilon \to 0} \phi(\mathbf{y}_1 \mid \mathbf{G}_\varepsilon) = \phi(\mathbf{y}_1 \mid \mathbf{G}). \end{aligned}$$

Here $\phi(\mathbf{y}_1 \mid \mathbf{H})$ is the conditional pdf of $\mathbf{Y}_1$ given $\mathbf{H}$, where $\mathbf{Y}_*$ is the *np-variate normal* random vector with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}_*$; i.e., $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ are i.i.d. *p-variate normal* random vectors with mean $\mathbf{0}$ and variance $\mathbf{\Sigma}$. Q.E.D.

The above theorem is a finite version of the previous conditional limit theorems about large deviation problems (see, e.g., [1–3]). It implies that a *normal* random vector is the nearest in the Kullback–Leibler sense to a *uniform* random vector under the sample covariance constraint.

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